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Differential Inequalities for Second- and Third-Order Equations*

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1. INTRODUCTION

Let $I \subset R$ be an interval. In this paper the existence on I of certain solutions to second- and third-order differential equations of the form

$$x'' = f(t, x, x') \tag{1}$$

and

$$x''' = f(t, x, x', x'') \tag{2}$$

is established. It is assumed that f is continuous, that solutions of initial value problems extend to I, and that appropriate boundary value problems have no more than one solution. The novelty of these results lies in the fact that the existence of a lower solution ϕ and an upper solution ψ is assumed whose functional values satisfy the opposite inequality from what is assumed in papers dealing with existence theorems for boundary value problems. The solutions obtained are "between" the functions ϕ and ψ just as they are in the theorems dealing with boundary value problems, except that the positions of ϕ and ψ are reversed.

For second-order equations the following conditions will be assumed, where $I \subset R$ is an interval.

(A_2) $f: I \times R^2 \to R$ is continuous.

(B_2) If x_1 and x_2 are solutions of the differential equation (1) on $[t_1, t_2] \subset I$ where $t_1 < t_2$ such that $x_1(t_i) = x_2(t_i)$ for $i = 1, 2$ then $x_1(t) = x_2(t)$ for $t \in [t_1, t_2]$.

(C_2) All solutions of all initial value problems for (1) extend throughout I.

For third-order equations, with $I \subset R$ an interval, the analogous conditions to be assumed are as follows.

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(A_3) $f: I \times R^3 \to R$ is continuous.

(B_3) If x_1 and x_2 are solutions of the differential equation (2) on $[t_1, t_3] \subset I$ where $t_1 < t_2 < t_3$ such that $x_1(t_i) = x_2(t_i)$ for $i = 1, 2, 3$ then $x_1(t) = x_2(t)$ for $t \in [t_1, t_3]$.

(C_3) All solutions of all initial value problems for (2) extend throughout I.

In numerous papers dealing with boundary value problems for second-order differential equations of the form (1) it is assumed that there exists an upper solution ψ and a lower solution ϕ of (1) on I with $\phi(t) \leqslant \psi(t)$ for $t \in I$. Such results, for example, may be found in [1–7, 9–22, 24, 25, 28–39, 41–45]. In this paper we assume the opposite inequality holds between ϕ and ψ (i.e., $\psi(t) \leqslant \phi(t)$ for $t \in I$) and are able to show, using conditions (A_2), (B_2), and (C_2), that there exists a solution of (1) on I which lies between ψ and ϕ.

We also treat third-order equations of the form (2) where again we assume inequalities hold between an upper solution ψ and a lower solution ϕ which are opposite from what is usually found in papers dealing with two- or three-point boundary value problems for (2), such as in [29, 36, 43, 45].

2. Second-Order Equations

We begin by establishing a result for upper and lower solutions which are equal at some point.

Lemma 2.1. *Let $I \subset R$ be an interval and assume that* (A_2), (B_2), *and* (C_2) *hold. Let $\phi, \psi \in C^2(I)$ be lower and upper solutions, respectively, of* (1) *on I with $\psi(t) \leqslant \phi(t)$ for $t \in I$ and $\psi(t_0) = \phi(t_0)$ for some $t_0 \in I$. Then there is a solution x of* (1) *on I with $\psi(t) \leqslant x(t) \leqslant \phi(t)$ for $t \in I$.*

Proof. If $\psi'(t_0) \neq \phi'(t_0)$ then t_0 must be an endpoint of I and any solution x of Eq. (1) satisfying the initial conditions

$$x(t_0) = \phi(t_0), \qquad x'(t_0) = (\phi'(t_0) + \psi'(t_0))/2$$

will satisfy the conclusion of the lemma by [40, Theorem 1, p. 1007].

In case $\psi'(t_0) = \phi'(t_0)$, we will show the existence of a solution z of (1) on $[t_0, +\infty) \cap I$ (if t_0 is not the right endpoint of I) satisfying the initial conditions

$$z(t_0) = \phi(t_0), \qquad z'(t_0) = \phi'(t_0)$$

and satisfying $\psi(t) \leqslant z(t) \leqslant \phi(t)$ for $t \in [t_0, +\infty) \cap I$. A similar argument applied to $(-\infty, t_0] \cap I$ (in case t_0 is not the left endpoint of I) then yields a solution y on $(-\infty, t_0] \cap I$ satisfying the initial conditions

$$y(t_0) = \phi(t_0), \qquad y'(t_0) = \phi'(t_0)$$

and satisfying $\psi(t) \leqslant y(t) \leqslant \phi(t)$ for $t \in (-\infty, t_0] \cap I$. If t_0 is the left endpoint of I, then z is the desired solution. If t_0 is the right endpoint of I, then y is the desired solution. If t_0 is neither the left endpoint of I nor the right endpoint of I, then x defined by $x(t) = y(t)$ for $t \in (-\infty, t_0] \cap I$ and $x(t) = z(t)$ for $t \in (t_0, +\infty) \cap I$ satisfies the conclusion of the lemma.

We assume now that t_0 is not the right endpoint of I. By [40, Theorem 4, p. 1011] it follows that if $\psi(t)$ and $\phi(t)$ are equal at two points t_1, t_2 with $t_1 < t_2$ then $\psi(t) = \phi(t)$ for $t_1 \leqslant t \leqslant t_2$. We may assume that neither ϕ nor ψ is a solution of (1) on $[t_0, +\infty) \cap I$ since then z could be chosen to be equal to ϕ or ψ, respectively. Thus there are points in $[t_0, +\infty) \cap I$ where $\phi(t)$ and $\psi(t)$ are not equal. Let $t_1 = \sup\{t: t \geqslant t_0, t \in I, \phi(t) = \psi(t)\}$. Then $\phi(t) = \psi(t)$ for $t_0 \leqslant t \leqslant t_1$ and $t_1 \in I$ is not the right endpoint of I. Let z_n be a solution of (1) on $[t_0, +\infty) \cap I$ satisfying the boundary conditions

$$z_n(t_0) = \phi(t_0), \qquad z_n(t_1 + 1/n) = (\phi(t_1 + 1/n) + \psi(t_1 + 1/n))/2$$

which exists, for n sufficiently large, by [24, Theorem 6.1, p. 344 and the last line of p. 346]. By [40, Theorem 1, p. 1007] it follows that $\psi(t) < z_n(t) < \phi(t)$ for $t \in [t_1 + 1/n, +\infty) \cap I$. It follows from [23, Theorem 3.2, p. 14], using the mean value theorem on $[t_1 + 1/n_1, t_1 + 1/n_2]$ (where $n_1 > n_2$ are chosen fixed but sufficiently large) to pick appropriate initial conditions, that there is a subsequence of $\{z_n\}$, which we denote again by $\{z_n\}$, and a solution z of (1) on $[t_0, +\infty) \cap I$ such that $z_n(t) \to z(t)$ and $z_n'(t) \to z'(t)$ as $n \to +\infty$, uniformly on compact subsets of $[t_0, +\infty) \cap I$. It follows that $\psi(t) \leqslant z(t) \leqslant \phi(t)$ for $t \in [t_1, +\infty) \cap I$. From this, we see that $\psi(t) \leqslant z(t) \leqslant \phi(t)$ for $t \in [t_0, +\infty) \cap I$, $z(t_0) = \phi(t_0)$ and $z'(t_0) = \phi'(t_0)$. This completes the proof of the lemma.

THEOREM 2.2. *Let $I \subset R$ be an interval and assume that* (A_2), (B_2), *and* (C_2) *hold. Let $\phi, \psi \in C^2(I)$ be lower and upper solutions, respectively, of* (1) *on I with $\psi(t) \leqslant \phi(t)$ for $t \in I$. Then there is a solution x of* (1) *on I with $\psi(t) \leqslant x(t) \leqslant \phi(t)$ for $t \in I$.*

Proof. If $\phi(t_0) = \psi(t_0)$ for some $t_0 \in I$, then the result follows from Lemma 2.1 so we may assume that $\psi(t) < \phi(t)$ for $t \in I$. It suffices to prove the theorem when $I = [a, b]$ is a closed and bounded interval since any other interval I can be written as a union of a nested sequence of closed bounded intervals I_n and the solutions x_n, known to exist on I_n and satisfy $\psi(t) \leqslant x_n(t) \leqslant \phi(t)$ for $t \in I_n$, would have a subsequence converging to the desired solution x on I.

For each fixed real number $\lambda \in [a, b]$, let $\{v_n\}$ and $\{y_n\}$ be sequences of solutions of (1) on $[a, b]$ satisfying the respective initial conditions

$$v_n(\lambda) = \psi(\lambda), \qquad v_n'(\lambda) = \psi'(\lambda) + 1/n \tag{3}$$

and

$$y_n(\lambda) = \psi(\lambda), \qquad y_n'(\lambda) = \psi'(\lambda) - 1/n. \tag{4}$$

By [23, Theorem 3.2, p. 14] there exist solutions v and y of (1) on $[a, b]$ which are limits of subsequences of $\{v_n\}$ and $\{y_n\}$, respectively, satisfying the initial conditions

$$v(\lambda) = \psi(\lambda), \qquad v'(\lambda) = \psi'(\lambda), \tag{5}$$

and

$$y(\lambda) = \psi(\lambda), \qquad y'(\lambda) = \psi'(\lambda). \tag{6}$$

We note that by [40, Theorem 1, p. 1007] it follows that $v_n(t) \geqslant \psi(t)$ for $\lambda \leqslant t \leqslant b$ and $y_n(t) \geqslant \psi(t)$ for $a \leqslant t \leqslant \lambda$. Thus $v(t) \geqslant \psi(t)$ for $\lambda \leqslant t \leqslant b$ and $y(t) \geqslant \psi(t)$ for $a \leqslant t \leqslant \lambda$. In fact, when $a \leqslant \lambda < b$, the solution v on $[\lambda, b]$ will be the unique right maximal solution of the initial value problem (1), (5) by [8, Theorem 1, p. 126] and [8, Theorem 3, p. 128].

We now define a solution x_λ of (1) on $[a, b]$ by $x_\lambda(t) = y(t)$ for $a \leqslant t \leqslant \lambda$ and $x_\lambda(t) = v(t)$ for $\lambda < t \leqslant b$. By [40, Theorem 1, p. 1007] it follows that $x_\lambda(t)$ cannot equal $\phi(t)$ at two points $t_1(\lambda)$, $t_2(\lambda) \in [a, b]$ with $t_1(\lambda) < \lambda < t_2(\lambda)$. We may assume that for each $\lambda \in [a, b]$ either $t_1(\lambda) < \lambda$ exists in $[a, b]$ with $x_\lambda(t_1(\lambda)) = \phi(t_1(\lambda))$ or else $t_2(\lambda) > \lambda$ exists in $[a, b]$ with $x_\lambda(t_2(\lambda)) = \phi(t_2(\lambda))$, for otherwise, for such a λ, x_λ would satisfy the conclusion of the theorem. Since for $\lambda = a$ the number $t_2(a)$ exists while for $\lambda = b$ the number $t_1(b)$ exists, there is a $\lambda_0 \in [a, b]$ and a sequence $\lambda_n \to \lambda_0$ such that either $t_1(\lambda_0)$ and $\{t_2(\lambda_n)\}$ exist or else $t_2(\lambda_0)$ and $\{t_1(\lambda_n)\}$ exist. Only the case where $t_1(\lambda_0)$ and $\{t_2(\lambda_n)\}$ exist will be treated, since the other case is similar. By [23, Theorem 3.2 p. 14] there is a solution z of (1) on $[a, b]$ which is the limit of a subsequence of $\{x_{\lambda_n}\}$. If we define $x(t)$ by $x(t) = x_{\lambda_0}(t)$ for $\lambda_0 \leqslant t \leqslant b$ and $x(t) = z(t)$ for $a \leqslant t \leqslant \lambda_0$, then x is the desired solution and the proof is complete.

We observe that for the equation $x'' = 0$ on R it is easy to find a lower solution ϕ and an upper solution ψ, neither of which are solutions of $x'' = 0$, satisfying $\psi(t) \leqslant \phi(t)$ on R. If we wished the opposite inequality to hold between ϕ and ψ, i.e., $\phi(t) \leqslant \psi(t)$ on R, it would follow that ϕ and ψ were each solutions of $x'' = 0$. See [42, Theorem 5.1, p. 579] for a result similar to Theorem 2.2 but with the inequality between ϕ and ψ reversed.

3. Third-Order Equations

It is not known whether a function $\phi \in C^3(I)$ which is a lower solution for (2) is necessarily a subfunction for (2) when (A_3), (B_3), and (C_3) hold [27, p. 181]. Thus we add an extra condition, (D_3), for third-order equations that was not used in proving the corresponding results for second-order equations.

(D_3) Either $\phi, \psi \in C^3(I)$ are strict lower and upper solutions, respectively, for (2) on I or else $f = f(t, y, z, w)$ in (2) satisfies a Lipschitz condition with respect to y, z, and w on compact subsets of $I \times R^3$.

Whenever we need to cite a result concerning superfunctions in this section we will just cite the analogous result for subfunctions, since usually only these results appear in the literature.

LEMMA 3.1. *Let $I \subset R$ be a bounded open interval, $\alpha \in I$, and assume that* (A_3), (B_3), *and* (C_3) *hold. Let $\phi, \psi \in C^3(I)$ be lower and upper solutions, respectively, of* (2) *on I with $\psi(t) \geqslant \phi(t)$ for $t \in (-\infty, \alpha] \cap I$ and $\psi(t) \leqslant \phi(t)$ for $t \in [\alpha, +\infty) \cap I$. Assume there is a point $t_0 > \alpha$, $t_0 \in I$, with $\phi(t_0) = \psi(t_0)$ and assume that* (D_3) *holds. Then there is a solution z of* (2) *on $[\alpha, +\infty) \cap I$ satisfying the boundary conditions*

$$z(\alpha) = \phi(\alpha), \qquad z(t_0) = \phi(t_0), \qquad z'(t_0) = \phi'(t_0)$$

and satisfying $\psi(t) \leqslant z(t) \leqslant \phi(t)$ for $t \in [\alpha, +\infty) \cap I$.

Proof. By [27, Remark 5.1, p. 193] it follows that if $\psi(t)$ and $\phi(t)$ are equal at some point $t_1 > t_0$ in I, then $\psi(t) = \phi(t)$ for $t_0 \leqslant t \leqslant t_1$ (in case the second of the two alternate hypotheses in (D_3) holds, [27, Remark 5.1] is still correct if the word strict is deleted). Similarly, if $\psi(t)$ and $\phi(t)$ are equal at some point t_2, $\alpha < t_2 < t_0$, then $\psi(t) = \phi(t)$ for $t_2 \leqslant t \leqslant t_0$. We may assume that neither ϕ nor ψ is a solution of (2) on $[\alpha, +\infty) \cap I$ since then z could be chosen to be equal to ϕ or ψ, respectively. Thus there are points in $[\alpha, +\infty) \cap I$ where $\phi(t)$ and $\psi(t)$ are not equal. Let $t_1 = \sup\{t \colon t \geqslant t_0, t \in I, \phi(t) = \psi(t)\}$ and $t_2 = \inf\{t \colon \alpha < t \leqslant t_0, \phi(t) = \psi(t)\}$. Then $\phi(t) = \psi(t)$ for $t_2 \leqslant t < t_1$ and either $t_2 > \alpha$ or else t_1 is not the right endpoint of I.

If t_1 is not the right endpoint of I, let z_n be a solution of (2) satisfying the boundary conditions

$$z_n(\alpha) = \phi(\alpha), \quad z_n(t_0) = \phi(t_0), \quad z_n(t_1 + 1/n) = (\phi(t_1 + 1/n) + \psi(t_1 + 1/n))/2.$$

That these solutions exist, for n chosen large enough that $t_1 + 1/n \in I$, follows from [26, Theorem 3, p. 48]. If t_1 is not the right endpoint of I, then $\psi(t) < z_n(t) < \phi(t)$ for $t \in [t_1 + 1/n, +\infty) \cap I$ by [27, Theorem 4.1, p. 191] and [27, Theorem 4.3, p. 192]. By [26, Theorem 1, p. 47] and [23, Theorem 3.2, p. 14] it follows that there is a solution z of (2) on $[\alpha, +\infty) \cap I$ and a subsequence of $\{z_n\}$, again denoted by $\{z_n\}$, such that $z_n^{(i)}(t) \to z^{(i)}(t)$, for $i = 0, 1, 2$, as $n \to +\infty$, uniformly on compact subintervals of $[\alpha, +\infty) \cap I$. It follows that $\psi(t) \leqslant z(t) \leqslant \phi(t)$ for $t \in [t_1, +\infty) \cap I$ and hence, by [27, Theorem 4.2, p. 192] and [27, Theorem 4.4, p. 193], we must have $\psi(t) \leqslant z(t) \leqslant \phi(t)$ for $t \in [\alpha, +\infty) \cap I$. Thus z would be the desired solution.

If t_1 is the right endpoint of I, let $\{\alpha_n\}$ be a monotone strictly increasing sequence of numbers converging to t_1 and let u_n be a solution of (2) satisfying the boundary conditions

$$u_n(\alpha) = \phi(\alpha), \qquad u_n(t_0) = \phi(t_0), \qquad u_n(\alpha_n) = \phi(\alpha_n).$$

That these solutions exist, for n chosen large enough that $t_0 < \alpha_n$, follows from [26, Theorem 3, p. 48]. By [27, Theorem 4.1, p. 191] and [27, Theorem 4.3, p. 192] it follows that $\psi(t) \leqslant u_n(t) \leqslant \phi(t)$ for $t \in [\alpha, \alpha_n]$. By [26, Theorem 1, p. 47] and [23, Theorem 3.2, p. 14] it follows that there is a solution u of (2) on $[\alpha, +\infty) \cap I$ and a subsequence of $\{u_n\}$, again denoted by $\{u_n\}$, such that $u_n^{(i)}(t) \to u^{(i)}(t)$ for $i = 0, 1, 2$, as $n \to +\infty$, uniformly on compact subintervals of $[\alpha, +\infty) \cap I$. It follows that $\psi(t) \leqslant u(t) \leqslant \phi(t)$ for $t \in [\alpha, +\infty) \cap I$ so that u satisfies the conclusion of the lemma.

LEMMA 3.2. *If the hypotheses of Lemma* 3.1 *hold, then there is a solution* y *of* (2) *on* $(-\infty, t_0] \cap I$ *satisfying the boundary conditions*

$$y(\alpha) = \phi(\alpha), \qquad y(t_0) = \phi(t_0), \qquad y'(t_0) = \phi'(t_0)$$

and satisfying $\psi(t) \geqslant y(t) \geqslant \phi(t)$ *for* $t \in (-\infty, \alpha] \cap I$ *and* $\psi(t) \leqslant y(t) \leqslant \phi(t)$ *for* $t \in [\alpha, t_0]$.

Proof. By [27, Remark 5.1, p. 193] it follows that if $\psi(t)$ and $\phi(t)$ are equal at some point $t_1 < \alpha$ in I, then $\psi(t) = \phi(t)$ for $t_1 \leqslant t \leqslant t_0$ (in case the second of the two alternate hypotheses in (D_3) holds, [27, Remark 5.1] is still correct if the word strict is deleted). Similarly if $\psi(t)$ and $\phi(t)$ are equal at some point t_2, $\alpha < t_2 < t_0$, then $\psi(t) = \phi(t)$ for $t_2 \leqslant t \leqslant t_0$. We may assume that neither ϕ nor ψ is a solution of (2) on $(-\infty, t_0] \cap I$ since then y could be chosen to be equal to ϕ or ψ, respectively. Thus there are points in $(-\infty, t_0] \cap I$ where $\phi(t)$ and $\psi(t)$ are not equal. Let $t_1 = \inf\{t\colon t \leqslant \alpha, t \in I, \phi(t) = \psi(t)\}$ and let $t_2 = \inf\{t\colon \alpha < t \leqslant t_0, \phi(t) = \psi(t)\}$. Then either $t_1 \leqslant \alpha$ and $\phi(t) = \psi(t)$ for $t_1 \leqslant t \leqslant t_0$ or else $t_1 = \alpha < t_2$ and $\phi(t) = \psi(t)$ for $t_2 \leqslant t \leqslant t_0$.

If $t_1 \leqslant \alpha$ and $\phi(t) = \psi(t)$ for $t_1 \leqslant t \leqslant t_0$, then we may assume that t_1 is not the left endpoint of I. Now let $\{y_n\}$ be a sequence of solutions of (2) on $(-\infty, t_0] \cap I$ satisfying the boundary conditions

$$y_n(t_1 - 1/n) = (\phi(t_1 - 1/n) + \psi(t_1 - 1/n))/2, \quad y_n(t_1) = \phi(t_1), \quad y_n(t_0) = \phi(t_0).$$

That these solutions exist, for n chosen large enough that $t_1 - 1/n \in I$, follows from [26, Theorem 3, p. 48]. Moreover, $\phi(t) < y_n(t) < \psi(t)$ for $t \in (-\infty, t_1 - 1/n] \cap I$ by [27, Theorem 4.1, p. 191] and [27, Theorem 4.3, p. 192]. By [26, Theorem 1, p. 47] and [23, Theorem 3.2, p. 14] it follows that there is a subsequence of $\{y_n\}$, again denoted by $\{y_n\}$, and a solution y of (2) on

$(-\infty, t_0] \cap I$ such that $y_n^{(i)}(t) \to y^{(i)}(t)$ for $i = 0, 1, 2$, as $n \to +\infty$, uniformly on compact subintervals of $(-\infty, t_0] \cap I$. It follows that $\phi(t) \leqslant y(t) \leqslant \psi(t)$ for $t \in (-\infty, t_1]$ and that $\phi(t) = y(t) = \psi(t)$ for $t \in [t_1, t_0]$ by [27, Theorem 4.2, p. 192] and [27, Theorem 4.4, p. 193]. Thus y would satisfy the conclusion of the lemma.

If $t_1 = \alpha < t_2$ and $\phi(t) = \psi(t)$ for $t_2 \leqslant t \leqslant t_0$, then we note that $\phi'(\alpha) > \psi'(\alpha)$ by [27, Remark 5.1, p. 193] (in case the second of the two alternate hypotheses in (D_3) holds, [27, Remark 5.1] is still correct if the word strict is deleted). Now let y be a solution of (2) satisfying the boundary conditions

$$y(\alpha) = \phi(\alpha), \qquad y(t_0) = \phi(t_0), \qquad y'(t_0) = \phi'(t_0).$$

That this solution exists follows from [26, Theorem 2, p. 47]. Moreover, $\psi(t) \leqslant y(t) \leqslant \phi(t)$ for $\alpha \leqslant t \leqslant t_0$ by [27, Theorem 4.2, p. 192] and [27, Theorem 4.4, p. 193]. It follows that $\psi'(\alpha) \leqslant y'(\alpha) \leqslant \phi'(\alpha)$. If in fact we have $\psi'(\alpha) < y'(\alpha) < \phi'(\alpha)$, then $\phi(t) < y(t) < \psi(t)$ for $t \in (-\infty, \alpha) \cap I$ by [27, Theorem 4.1, p. 191] and [27, Theorem 4.3, p. 192], and then y satisfies the conclusion of the lemma. The remaining possibility is that either $\psi'(\alpha) < y'(\alpha) = \phi'(\alpha)$ or else $\psi'(\alpha) = y'(\alpha) < \phi'(\alpha)$. We treat only the first case since the second one is similar. We note that if $y'(\alpha) = \phi'(\alpha)$, then $y(t) = \phi(t)$ for $t \in [\alpha, t_0]$ by [27, Theorem 4.2, p. 192], [27, Theorem 4.4, p. 193], and the fact that we already have $y(t) \leqslant \phi(t)$ for $t \in [\alpha, t_0]$. Let v_n be a solution of (2) satisfying the initial conditions

$$v_n(\alpha) = \phi(\alpha), \qquad v_n'(\alpha) = \phi'(\alpha), \qquad v_n''(\alpha) = \phi''(\alpha) + 1/n.$$

By [27, Theorem 4.2, p. 192] and [27, Theorem 4.4, p. 193] it follows that $\phi(t) < v_n(t)$ for $t \in (-\infty, \alpha) \cap I$. By [23, Theorem 3.2, p. 14] there is a subsequence of $\{v_n\}$, again denoted by $\{v_n\}$, and a solution v of (2) such that $v_n^{(i)}(t) \to v^{(i)}(t)$ for $i = 0, 1, 2$, as $n \to +\infty$, uniformly on compact subsets of $(-\infty, t_0] \cap I$. The function z defined by $z(t) = v(t)$ for $t \in (-\infty, \alpha]$ and $z(t) = y(t)$ for $t \in (\alpha, t_0]$ is a solution of (2) on $(-\infty, t_0] \cap I$ that satisfies $z(t) < \psi(t)$ for $t \in (-\infty, \alpha) \cap I$ by [27, Theorem 4.1, p. 191] and [27, Theorem 4.3, p. 192], and hence satisfies the conclusion of the lemma.

LEMMA 3.3. *Let $I \subset R$ be a bounded open interval, $\alpha \in I$, and assume that* (A_3), (B_3), *and* (C_3) *hold. Let $\phi, \psi \in C^3(I)$ be lower and upper solutions, respectively, of* (2) *on I with $\psi(t) > \phi(t)$ for $t \in (-\infty, \alpha) \cap I$ and $\psi(t) < \phi(t)$ for $t \in (\alpha, +\infty) \cap I$. Assume that $\phi'(\alpha) = \psi'(\alpha)$ and that* (D_3) *holds. Then there is a solution x of* (2) *on I satisfying $\psi(t) \geqslant x(t) \geqslant \phi(t)$ for $t \in (-\infty, \alpha] \cap I$ and $\psi(t) \leqslant x(t) \leqslant \phi(t)$ for $t \in [\alpha, +\infty) \cap I$.*

Proof. It follows from the hypotheses that $\psi''(\alpha) = \phi''(\alpha)$. Let u_n be a solution of (2) satisfying the boundary conditions

$$u_n(\alpha) = \phi(\alpha), \qquad u_n'(\alpha) = \phi'(\alpha), \qquad u_n(\alpha_n) = (\phi(\alpha_n) + \psi(\alpha_n))/2,$$

where $\{\alpha_n\}$ is a monotone strictly decreasing sequence converging to α [26, Theorem 2, p. 47]. By [27, Theorem 4.2, p. 192] and [27, Theorem 4.4, p. 193], $\psi(t) < u_n(t) < \phi(t)$ for $t \in [\alpha_n, +\infty) \cap I$. By [26, Theorem 1, p. 47] and [23, Theorem 3.2, p. 14] there is a subsequence of $\{u_n\}$, and a solution u of (2) on I, such that $u_n^{(i)}(t) \to u^{(i)}(t)$ for $i = 0, 1, 2$, as $n \to +\infty$, uniformly on compact subintervals of I. Thus $\psi(t) \leqslant u(t) \leqslant \phi(t)$ for $t \in [\alpha, +\infty) \cap I$ and $u(\alpha) = \phi(\alpha)$, $u'(\alpha) = \phi'(\alpha)$, $u''(\alpha) = \phi''(\alpha)$. A similar argument applied to the interval $(-\infty, \alpha] \cap I$ would yield a solution v of (2) satisfying $\phi(t) \leqslant v(t) \leqslant \psi(t)$ for $t \in (-\infty, \alpha] \cap I$ and $v(\alpha) = \phi(\alpha)$, $v'(\alpha) = \phi'(\alpha)$, $v''(\alpha) = \phi''(\alpha)$. The solution x of (2) on I defined by $x(t) = v(t)$ for $t \in (-\infty, \alpha] \cap I$ and $x(t) = u(t)$ for $t \in (\alpha, +\infty) \cap I$ would then satisfy the conclusion of the lemma.

LEMMA 3.4. *Let $I \subset R$ be a bounded open interval, $\alpha \in I$, and assume that* (A_3), (B_3), *and* (C_3) *hold. Let ϕ, $\psi \in C^3(I)$ be lower and upper solutions, respectively, of* (2) *on I with $\psi(t) > \phi(t)$ for $t \in (-\infty, \alpha) \cap I$ and $\psi(t) < \phi(t)$ for $t \in (\alpha, +\infty) \cap I$. Assume that $\phi'(\alpha) > \psi'(\alpha)$ and that* (D_3) *holds. Then for each real number λ, $\lambda > \alpha$, in I there is a solution x_λ of* (2) *on I satisfying $x_\lambda(\alpha) = \psi(\alpha)$, $x_\lambda(\lambda) = \psi(\lambda)$, $x_\lambda'(\lambda) = \psi'(\lambda)$, $\psi(t) \geqslant x_\lambda(t)$ for $t \in (-\infty, \alpha] \cap I$, $\psi(t) \leqslant x_\lambda(t)$ for $t \in [\alpha, +\infty) \cap I$ and either $\phi(t) \leqslant x_\lambda(t)$ for $t \in (-\infty, \alpha] \cap I$ and $x_\lambda(t) \leqslant \phi(t)$ for $t \in [\alpha, \lambda]$ or else $x_\lambda(t) \leqslant \phi(t)$ for $t \in [\lambda, +\infty) \cap I$.*

Proof. For each fixed real number $\lambda \in I$, $\lambda > \alpha$, let w be a solution of (2) satisfying the boundary conditions

$$w(\alpha) = \psi(\alpha), \qquad w(\lambda) = \psi(\lambda), \qquad w'(\lambda) = \psi'(\lambda),$$

which exists by [26, Theorem 2, p. 47]. Then $\psi(t) \leqslant w(t)$ for $t \in [\alpha, \lambda]$ by [27, Theorem 4.2, p. 192] and [27, Theorem 4.4, p. 193]. We note that the solution w satisfies $w''(\lambda) \geqslant \psi''(\lambda)$.

Now let z_n be a solution of (2) on I satisfying the initial conditions

$$z_n(\lambda) = \psi(\lambda), \qquad z_n'(\lambda) = \psi'(\lambda), \qquad z_n''(\lambda) = w''(\lambda) + 1/n.$$

Then $z_n(t) > \psi(t)$ for $t \in (\lambda, +\infty) \cap I$ by [27, Theorem 4.2, p. 192] and [27, Theorem 4.4, p. 193]. There is a subsequence of $\{z_n\}$, again denoted by $\{z_n\}$ and a solution z of (2) on I such that $z_n^{(i)}(t) \to z^{(i)}(t)$ for $i = 0, 1, 2$, as $n \to +\infty$ Thus $\psi(t) \leqslant z(t)$ for $t \in [\lambda, +\infty) \cap I$.

Let u_n be a solution of (2) on I satisfying the initial conditions

$$u_n(\alpha) = w(\alpha), \qquad u_n'(\alpha) = w'(\alpha), \qquad u_n''(\alpha) = w''(\alpha) - 1/n.$$

There is a subsequence of $\{u_n\}$, again denoted by $\{u_n\}$, and a solution u of (2) on I such that $u_n^{(i)}(t) \to u^{(i)}(t)$ for $i = 0, 1, 2$, as $n \to +\infty$. Note that if $w'(\alpha) = \psi'(\alpha)$, then $w(t) = \psi(t)$ for $\alpha \leqslant t \leqslant \lambda$ by [27, Theorem 4.2, p. 192] and [27, Theorem 4.4, p. 193] so that $u_n(t) < \psi(t)$ for $t \in (-\infty, \alpha) \cap I$ by

[27, Theorem 4.2, p. 192] and [27, Theorem 4.4, p. 193], and hence $u(t) \leqslant \psi(t)$ for $t \in (-\infty, \alpha] \cap I$. On the other hand, if $w'(\alpha) > \psi'(\alpha)$, then the function v defined by $v(t) = u(t)$ for $t \in (-\infty, \alpha] \cap I$ and $v(t) = w(t)$ for $t \in (\alpha, +\infty) \cap I$ would be a solution of (2) on I and v would have to satisfy $v(t) \leqslant \psi(t)$ for $t \in (-\infty, \alpha] \cap I$ by [27, Theorem 4.1, p. 191] and [27, Theorem 4.3, p. 192]. Thus in either case we would have $u(t) \leqslant \psi(t)$ for $t \in (-\infty, \alpha] \cap I$.

The function x defined by $x(t) = u(t)$ for $t \in (-\infty, \alpha] \cap I$, $x(t) = w(t)$ for $t \in (\alpha, \lambda]$ and $x(t) = z(t)$ for $t \in (\lambda, +\infty) \cap I$ now satisfies the conclusion of the lemma, since by [27, Theorems 4.1, 4.2, 4.3, and 4.4, pp. 191–193], either $x(t) \leqslant \phi(t)$ for $t \in [\lambda, +\infty) \cap I$, or else $\phi(t) \leqslant x(t)$ for $t \in (-\infty, \alpha] \cap I$ and $x(t) \leqslant \phi(t)$ for $t \in [\alpha, \lambda]$.

A result similar to Lemma 3.4 but for values of $\lambda < \alpha$ is stated next.

LEMMA 3.5. *Let $I \subset R$ be a bounded open interval, $\alpha \in I$, and assume that* (A_3), (B_3), *and* (C_3) *hold. Let $\phi, \psi \in C^3(I)$ be lower and upper solutions respectively, of* (2) *on I with $\psi(t) > \phi(t)$ for $t \in (-\infty, \alpha) \cap I$ and $\psi(t) < \phi(t)$ for $t \in (\alpha, +\infty) \cap I$. Assume that $\phi'(\alpha) > \psi'(\alpha)$ and that* (D_3) *holds. Then for each real number λ, $\lambda < \alpha$, in I there is a solution x_λ of* (2) *on I satisfying $x_\lambda(\alpha) = \psi(\alpha)$, $x_\lambda(\lambda) = \psi(\lambda)$, $x_\lambda'(\lambda) = \psi'(\lambda)$, $x_\lambda(t) \leqslant \psi(t)$ for $t \in (-\infty, \alpha] \cap I$, $\psi(t) \leqslant x_\lambda(t)$ for $t \in [\alpha, +\infty) \cap I$ and either $\phi(t) \leqslant x_\lambda(t)$ for $t \in (-\infty, \lambda] \cap I$ or else $\phi(t) \leqslant x_\lambda(t)$ for $t \in [\lambda, \alpha]$ and $x_\lambda(t) \leqslant \phi(t)$ for $t \in [\alpha, +\infty) \cap I$.*

Proof. The proof is similar to the proof of Lemma 3.4 so it is omitted. The next theorem is the main result of this section.

THEOREM 3.6. *Let $I \subset R$ be an interval, $\alpha \in I^o$, and assume that* (A_3), (B_3), *and* (C_3) *hold. Let $\phi, \psi \in C^3(I)$ be lower and upper solutions, respectively, of* (2) *on I with $\psi(t) \geqslant \phi(t)$ for $t \in (-\infty, \alpha] \cap I$ and $\psi(t) \leqslant \phi(t)$ for $t \in [\alpha, +\infty) \cap I$. If* (D_3) *holds then there is a solution x of* (2) *on I with $\psi(t) \geqslant x(t) \geqslant \phi(t)$ for $t \in (-\infty, \alpha] \cap I$ and $\psi(t) \leqslant x(t) \leqslant \phi(t)$ for $t \in [\alpha, +\infty) \cap I$.*

Proof. It suffices to prove the theorem when I is a bounded open interval since the interior of any interval I, with $\alpha \in I^o$, can be written as a union of a nested sequence of bounded open intervals I_n, containing α, and the solutions x_n, known to exist on I_n and satisfy $\phi(t) \leqslant x_n(t) \leqslant \psi(t)$ for $t \in (-\infty, \alpha] \cap I_n$ and $\psi(t) \leqslant x_n(t) \leqslant \phi(t)$ for $t \in [\alpha, +\infty) \cap I_n$, would have a subsequence converging to the desired solution x on I by [26, Theorem 1, p. 47] and [23, Theorem 3.2, p. 14].

If $\phi(t_0) = \psi(t_0)$ for some $t_0 \in I$, $t_0 \neq \alpha$, then assume $t_0 > \alpha$ since the other case is similar and let the solutions z and y be the solutions of (2) whose existence is assured by Lemma 3.1 and Lemma 3.2, respectively. Define x by $x(t) = y(t)$ for $t \in (-\infty, t_0] \cap I$ and $x(t) = z(t)$ for $t \in (t_0, +\infty) \cap I$. Since $z(t) = y(t)$ for $t \in [\alpha, t_0]$ by [27, Theorem 3.1, p. 184], it follows that x is a solution of (2) on I which would satisfy the conclusion of the theorem.

Assume now that $\phi(t) \neq \psi(t)$ for $t \in I$, $t \neq \alpha$. We may also assume that $\phi'(\alpha) > \psi'(\alpha)$ since otherwise the conclusion follows from Lemma 3.3. For each fixed real number $\lambda \in I$, $\lambda \neq \alpha$, let x_λ be the solution of (2) on I given by Lemma 3.4 or Lemma 3.5, whichever is appropriate.

Let the region M be defined by

$$M = \{(t, x): t \in I, \phi(t) \leqslant x \leqslant \psi(t) \text{ for } t \in (-\infty, \alpha] \cap I, \psi(t) \leqslant x \leqslant \phi(t)$$
$$\text{for } t \in (\alpha, +\infty) \cap I\}.$$

It follows from Lemmas 3.4 and 3.5 that for each $\lambda \neq \alpha$, either $L(\lambda)$ or $R(\lambda)$ holds for x_λ where these are the properties

$$L(\lambda) \qquad (t, x_\lambda(t)) \in M \qquad \text{for} \quad t \in (-\infty, \lambda] \cap I$$

and

$$R(\lambda) \qquad (t, x_\lambda(t)) \in M \qquad \text{for} \quad t \in [\lambda, +\infty) \cap I.$$

If $L(\lambda)$ holds for all $\lambda \in I$, $\lambda \neq \alpha$, then we choose $\alpha(n) > \alpha$ to be a monotone strictly increasing sequence converging to the right endpoint of I. The corresponding sequence $\{x_{\alpha(n)}\}$ of solutions of (2) given by Lemma 3.4 would, by [26, Theorem 1, p. 47] and [23, Theorem 3.2, p. 14], have a subsequence, again denoted by $\{x_{\alpha(n)}\}$, such that $x_{\alpha(n)}^{(i)}(t) \to x^{(i)}(t)$ for $i = 0, 1, 2$, as $n \to +\infty$, where x would be a solution of (2) on I satisfying the conclusion of the theorem.

If $R(\lambda)$ holds for all $\lambda \in I$, $\lambda \neq \alpha$, then an argument similar to the one just given shows that the conclusion of the theorem holds.

If there are values λ_1, λ_2 of $\lambda \neq \alpha$ in I such that $R(\lambda_1)$ and $L(\lambda_2)$ hold then there is a λ_0 in the closed interval determined by λ_1 and λ_2 and two monotone sequences of opposite monotonicity (not necessarily strictly monotone unless $\lambda_0 = \alpha$) $\alpha(n)$ and $\beta(n)$ converging to λ_0 such that $L(\alpha(n))$ and $R(\beta(n))$ hold for $n = 1, 2, \ldots$. By [26, Theorem 1, p. 47] and [23, Theorem 3.2, p. 14] there are solutions u and v of (2) on I and subsequences of $\{x_{\alpha(n)}\}$ and $\{x_{\beta(n)}\}$, again denoted, respectively, by $\{x_{\alpha(n)}\}$ and $\{x_{\beta(n)}\}$, such that $x_{\alpha(n)}^{(i)}(t) \to u^{(i)}(t)$ and $x_{\beta(n)}^{(i)}(t) \to v^{(i)}(t)$ for $i = 0, 1, 2$, as $n \to +\infty$, uniformly on compact subintervals of I. It follows that $(t, u(t)) \in M$ for $t \in (-\infty, \lambda_0] \cap I$ and that $(t, v(t)) \in M$ for $t \in [\lambda_0, +\infty) \cap I$.

If $\lambda_0 \neq \alpha$, then $u(\alpha) = \psi(\alpha) = v(\alpha)$ and $u(\lambda_0) = \psi(\lambda_0) = v(\lambda_0)$. Moreover, $u'(\lambda_0) = \psi'(\lambda_0) = v'(\lambda_0)$ so that by [27, Theorem 3.1, p. 184] we would have $u(t) = v(t)$ for t in the interval determined by α and λ_0. Thus x defined by $x(t) = u(t)$ for $t \in (-\infty, \lambda_0] \cap I$ and $x(t) = v(t)$ for $t \in (\lambda_0, +\infty) \cap I$ would be a solution of (2) on I satisfying the conclusion of the theorem.

If $\lambda_0 = \alpha$, we treat the case where $\alpha(n)$ is strictly monotone increasing and $\beta(n)$ is strictly monotone decreasing. The other case, where the monotonicity of $\alpha(n)$ and $\beta(n)$ is reversed, is similar so it is omitted. Let $\lambda_0 = \alpha$ and $\alpha(n)$ be strictly monotone increasing to λ_0 while $\beta(n)$ is strictly monotone decreasing to λ_0. Then $u(\alpha) = \psi(\alpha) = v(\alpha)$ and $u'(\alpha) = \psi'(\alpha) = v'(\alpha)$ since $x'_{\alpha(n)}(\alpha(n)) = \psi'(\alpha(n))$ and $x'_{\beta(n)}(\beta(n)) = \psi'(\beta(n))$. Moreover, $x''_{\alpha(n)}(\alpha(n)) \leqslant \psi''(\alpha(n))$ and $x''_{\beta(n)}(\beta(n)) \geqslant \psi''(\beta(n))$ so that $u''(\alpha) \leqslant \psi''(\alpha) \leqslant v''(\alpha)$. On the other hand there are points t_n, τ_n with $t_n \in [\alpha(n), \alpha]$ and $\tau_n \in [\alpha, \beta(n)]$ such that $x''_{\alpha(n)}(t_n) \geqslant \psi''(t_n)$ and $x''_{\beta(n)}(\tau_n) \leqslant \psi''(\tau_n)$. It follows that $v''(\alpha) \leqslant \psi''(\alpha) \leqslant u''(\alpha)$ so that $u''(\alpha) = \psi''(\alpha) = v''(\alpha)$. From this we see that the function x defined by $x(t) = u(t)$ for $t \in (-\infty, \alpha] \cap I$ and $x(t) = v(t)$ for $t \in (\alpha, +\infty) \cap I$ would be a solution of (2) on I satisfying the conclusion of the theorem.

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